New versions of q-surface pencil in Euclidean 3-space

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Abstract

In this paper, the q-surface pencil is studied in Euclidean 3-space. By using q-frame in Euclidean space, Firstly, we define q-surface pencil. Then, we give the necessary and sufficient condition for a curve to be a geodesic curve and to be an asymptotic curve on a q-surface pencil. Then, we study this subject for an offset q-surface pencil.

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1 Introduction

Geodesic on a surface corresponds to the shortest path between any two points on the surface. When one flattens a developable surface into a planar shape without distortion, any geodesic on the surface will be mapped to a straight line in the planar shape. Therefore, a good algorithm which is needed to flatten a non-developable surface with as little distortion as possible must preserve the geodesic curvatures on the surface, [16].

A geodesic on a surface is a parametric curve on the surface such that the acceleration vector on every point of the curve is orthogonal to the surface. It is well-known that the tangent vector field of a parametric curve on a surface is parallel in Levi-Civita sense if and only if this curve is a geodesic on the surface. A geodesic on a surface can also be defined as a curve on the surface with zero geodesic curvature, [10]. Additionally, there are many works related to surface pencil [1-3,8].

Geodesic curves play an important role in many industrial applications, such as tent manufacturing and textile manufacturing. Most existing work on geodesics is finding geodesics on a given surface. However, the reverse problem that is how to characterize the surfaces that possess a given curve as a common geodesic is also studied. Shoe design can be given as an engineering application of this problem. The characteristic curve of a woman shoe which is called the girth is wanted to be a geodesic on the shoe surface, [16].

One important concept in differential geometry of curves and surfaces is also asymptotic curves. An asymptotic curve on a surface is a curve such that its tangent vector field is an asymptotic direction. Along an asymptotic direction, the surface never makes twists from its tangent plane. Furthermore, the Gaussian curvature of the surface is never positive along an asymptotic direction. Moreover, the acceleration vector of the asymptotic curve on the surface is tangent to the surface, [4]. Additionally, there are many works related to this surfaces [11-14].

In this paper, we study q-surface pencil and its offset. Given a space curve with the q-frame, we obtain the conditions for this curve to be a common geodesic and to be a common asymptotic curve on a q-surface pencil. Then we give answers to the same problem for the offset q-surface pencil.

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2 Preliminaries

In this section, we present the Frenet frame and the q-frame along a space curve. Also, we give some geometric properties for these frames.

Let $\alpha(s)$ be a space curve parameterized with arc-length in $\mathbb{R}^3$. The Frenet frame of $\alpha(s)$ is $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$, where the vector fields are given as

$$\begin{aligned}
\mathbf{T}(s) &= \alpha'(s), \\
\mathbf{N}(s) &= \frac{\alpha''(s)}{\|\alpha''(s)\|}, \\
\mathbf{B}(s) &= \mathbf{T}(s)\times\mathbf{N}(s).
\end{aligned}$$

Here, $\mathbf{T}(s)$ is called the tangent vector field, $\mathbf{N}(s)$ is called the principal normal vector field and $\mathbf{B}(s)$ is called the binormal vector field of the curve $\alpha(s)$.

The curvature $\kappa(s)$ and the torsion $\tau(s)$ of the curve $\alpha(s)$ are given by

$$\begin{aligned}
\kappa(s) &= \|\alpha''(s)\|, \\
\tau(s) &= \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\|\alpha''(s)\|^2}.
\end{aligned}$$

The q-frame of a space curve $\alpha(s)$ which is parameterized with arc-length is $\{\mathbf{t}_q(s), \mathbf{n}_q(s), \mathbf{b}_q(s)\}$, where the vector fields are given as

$$\begin{aligned}
\mathbf{t}_q(s) &= \mathbf{T}(s), \\
\mathbf{n}_q(s) &= \frac{\mathbf{T}(s)\times\overrightarrow{\mathbf{k}}}{\left\|\mathbf{T}(s)\times\overrightarrow{\mathbf{k}}\right\|}, \\
\mathbf{b}_q(s) &= \mathbf{T}(s)\times\mathbf{n}_q(s).
\end{aligned}$$

Here, $\overrightarrow{\mathbf{k}}$ is the projection vector which can be chosen as $\overrightarrow{\mathbf{k}} = (1,0,0)$ or $\overrightarrow{\mathbf{k}} = (0,1,0)$ or $\overrightarrow{\mathbf{k}} = (0,0,1)$. In this paper, we choose the projection vector $\overrightarrow{\mathbf{k}} = (0,0,1)$. $\mathbf{n}_q(s)$ and $\mathbf{b}_q(s)$ are called the quasi normal vector field and the quasi binormal vector field of the curve $\alpha(s)$, respectively, [5].

Let $\theta(s)$ be the angle between the principal normal vector field $\mathbf{N}(s)$ and the quasi normal vector field $\mathbf{n}_q(s)$. The quasi formulas are given by

$$\frac{d}{ds}\begin{bmatrix} \mathbf{t}_q(s) \\ \mathbf{n}_q(s) \\ \mathbf{b}_q(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & k_3(s) \\ -k_2(s) & -k_3(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_q(s) \\ \mathbf{n}_q(s) \\ \mathbf{b}_q(s) \end{bmatrix},$$

where $k_i(s)$ are called the quasi curvatures $(1 \leq i \leq 3)$ which are given by

$$\begin{aligned}
k_1(s) &= \kappa(s)\cos\theta(s) = \left\langle \mathbf{t}_q'(s), \mathbf{n}_q(s) \right\rangle, \\
k_2(s) &= -\kappa(s)\sin\theta(s) = \left\langle \mathbf{t}_q'(s), \mathbf{b}_q(s) \right\rangle, \\
k_3(s) &= \theta'(s) + \tau(s) = -\left\langle \mathbf{n}_q(s), \mathbf{b}_q'(s) \right\rangle.
\end{aligned}$$

The relationship between the Frenet frame and the q-frame is given by, [6]. Additionally, there are many works related to q-frame, [7,15].

$$\begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & -\sin\theta(s) \\ 0 & \sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} \mathbf{t}_q(s) \\ \mathbf{n}_q(s) \\ \mathbf{b}_q(s) \end{bmatrix}.$$

3 q-Surface pencil with a common geodesic curve

We can define q-surface pencil similar to the definition of surface pencil in [9], as follows:

Definition 3.1. q-surface pencil possessing $\alpha(s)$ as a common curve is given by

$$S(s,t) = \alpha(s) + x_1(s,t)\mathbf{t}_q(s) + x_2(s,t)\mathbf{n}_q(s) + x_3(s,t)\mathbf{b}_q(s), \tag{3.1}$$

$0 \leq s \leq L$, $0 \leq t \leq T$, where $x_1(s,t)$, $x_2(s,t)$ and $x_3(s,t)$ are C^2 functions. The curve $\alpha(s)$ is a common parametric curve on the q-surface pencil $S(s,t)$ if

$$x_1(s,t_0) = x_2(s,t_0) = x_3(s,t_0) = 0.$$

Theorem 3.2. Let $\alpha(s)$ be a parametric curve on the q-surface pencil $S(s,t)$ which is given as (3.1). Then, $\alpha(s)$ is a geodesic curve on the surface $S(s,t)$ if and only if

$$\mu_1(s,t_0) = 0,\ \mu_2(s,t_0) = k\cos\theta(s),\ \mu_3(s,t_0) = -k\sin\theta(s), \tag{3.2}$$

where $m(s,t_0) = \mu_1(s,t_0)\mathbf{t}_q(s) + \mu_2(s,t_0)\mathbf{n}_q(s) + \mu_3(s,t_0)\mathbf{b}_q(s)$ is the normal vector field of $S(s,t)$ along the curve $\alpha(s)$, $\theta(s)$ is the angle between the principal normal vector field $\mathbf{N}(s)$ and the quasi normal vector field $\mathbf{n}_q(s)$ of the curve $\alpha(s)$ and $k \in \mathbb{R}$, $k \neq 0$.

Proof. The normal vector field of $S(s,t)$ is

$$\begin{aligned} m(s,t) &= \frac{\partial S}{\partial s}(s,t) \times \frac{\partial S}{\partial t}(s,t) \\ &= \mu_1(s,t)\mathbf{t}_q(s) + \mu_2(s,t)\mathbf{n}_q(s) + \mu_3(s,t)\mathbf{b}_q(s), \end{aligned}$$

where

$$\begin{aligned} \mu_1(s,t) &= (\frac{\partial x_3}{\partial t}(s,t)k_1(s) - \frac{\partial x_2}{\partial t}(s,t)k_2(s))x_1(s,t) \\ &\quad -(\frac{\partial x_3}{\partial t}(s,t)x_3(s,t) + \frac{\partial x_2}{\partial t}(s,t)x_2(s,t))k_3(s) \\ &\quad +\frac{\partial x_3}{\partial t}(s,t)\frac{\partial x_2}{\partial s}(s,t) - \frac{\partial x_2}{\partial t}(s,t)\frac{\partial x_3}{\partial s}(s,t), \\ \mu_2(s,t) &= (\frac{\partial x_1}{\partial t}(s,t)k_3(s) + \frac{\partial x_3}{\partial t}(s,t)k_1(s))x_2(s,t) \\ &\quad +(\frac{\partial x_3}{\partial t}(s,t)x_3(s,t) + \frac{\partial x_1}{\partial t}(s,t)x_1(s,t))k_2(s) \\ &\quad +\frac{\partial x_1}{\partial t}(s,t)\frac{\partial x_3}{\partial s}(s,t) - \frac{\partial x_3}{\partial t}(s,t)\frac{\partial x_1}{\partial s}(s,t) - \frac{\partial x_3}{\partial t}(s,t), \end{aligned}$$

$$
\begin{aligned}
\mu_3(s,t) &= (\frac{\partial x_1}{\partial t}(s,t)k_3(s) - \frac{\partial x_2}{\partial t}(s,t)k_2(s))x_3(s,t) \\
&\quad -(\frac{\partial x_2}{\partial t}(s,t)x_2(s,t) + \frac{\partial x_1}{\partial t}(s,t)x_1(s,t))k_1(s) \\
&\quad +\frac{\partial x_2}{\partial t}(s,t)\frac{\partial x_1}{\partial s}(s,t) - \frac{\partial x_1}{\partial t}(s,t)\frac{\partial x_2}{\partial s}(s,t) + \frac{\partial x_2}{\partial t}(s,t).
\end{aligned}
$$

Along the curve $\alpha(s)$, we can write

$$m(s,t_0) = \mu_1(s,t_0)\mathbf{t}_q(s) + \mu_2(s,t_0)\mathbf{n}_q(s) + \mu_3(s,t_0)\mathbf{b}_q(s),$$

where

$$
\begin{aligned}
\mu_1(s,t_0) &= \frac{\partial x_3}{\partial t}(s,t_0)\frac{\partial x_2}{\partial s}(s,t_0) - \frac{\partial x_2}{\partial t}(s,t_0)\frac{\partial x_3}{\partial s}(s,t_0), \\
\mu_2(s,t_0) &= -(1+\frac{\partial x_1}{\partial s}(s,t_0))\frac{\partial x_3}{\partial t}(s,t_0) - \frac{\partial x_3}{\partial t}(s,t_0)\frac{\partial x_1}{\partial s}(s,t_0), \\
\mu_3(s,t_0) &= (1+\frac{\partial x_1}{\partial s}(s,t_0))\frac{\partial x_2}{\partial t}(s,t_0) - \frac{\partial x_1}{\partial t}(s,t_0)\frac{\partial x_2}{\partial s}(s,t_0).
\end{aligned}
$$

Since $\alpha(s)$ is a parametric curve on the q-surface pencil $S(s,t)$, we can write

$$x_1(s,t_0) = x_2(s,t_0) = x_3(s,t_0) = 0.$$

We know that $\alpha(s)$ is a geodesic curve on the surface $S(s,t)$ if and only if the normal vector field $m(s,t_0)$ of the surface $S(s,t)$ along the curve $\alpha(s)$ is parallel to the principal normal vector field $\mathbf{N}(s)$ of the curve $\alpha(s)$, that is, $m(s,t_0)\|\mathbf{N}(s)$, [16]. Using the relationship between the Frenet frame and the q-frame, we obtain

$$
\begin{aligned}
m(s,t_0)\|\mathbf{N}(s) &\Leftrightarrow m(s,t_0)\|(\cos\theta(s)\mathbf{n}_q(s) - \sin\theta(s)\mathbf{b}_q(s)) \\
&\Leftrightarrow m(s,t_0) = k\cos\theta(s)\mathbf{n}_q(s) - k\sin\theta(s)\mathbf{b}_q(s) \\
&\Leftrightarrow \mu_1(s,t_0) = 0,\ \mu_2(s,t_0) = k\cos\theta(s),\ \mu_3(s,t_0) = -k\sin\theta(s).
\end{aligned}
$$

Q.E.D.

Corollary 3.3. Let $\alpha(s)$ be a parametric curve on the q-surface pencil $S(s,t)$ which is given as (3.1). Then, $\alpha(s)$ is a geodesic curve on the surface $S(s,t)$ if and only if

$$(\frac{\partial x_2}{\partial t}(s,t_0))^2 + (\frac{\partial x_3}{\partial t}(s,t_0))^2 = k^2, \tag{3.3}$$

where $k \in \mathbb{R}$, $k \neq 0$.

Proof. Since $\alpha(s)$ is a parametric curve on the q-surface pencil $S(s,t)$, we can write

$$x_1(s,t_0) = x_2(s,t_0) = x_3(s,t_0) = 0.$$

With the help of the definition of partial differentiation, we get

$$\frac{\partial x_1}{\partial s}(s,t_0) = \frac{\partial x_2}{\partial s}(s,t_0) = \frac{\partial x_3}{\partial s}(s,t_0) = 0.$$

So, we get $\mu_1(s,t_0)=0$, $\mu_2(s,t_0)=-\frac{\partial x_3}{\partial t}(s,t_0)$ and $\mu_3(s,t_0)=\frac{\partial x_2}{\partial t}(s,t_0)$. From Theorem 3.2, we know that $\alpha(s)$ is a geodesic curve on the surface $S(s,t)$ if and only if

$$\mu_1(s,t_0)=0,\ \mu_2(s,t_0)=k\cos\theta(s),\ \mu_3(s,t_0)=-k\sin\theta(s),$$

where $k\in\mathbb{R}$, $k\neq 0$. Therefore, we obtain $\alpha(s)$ is a geodesic curve on the surface $S(s,t)$ if and only if

$$(\frac{\partial x_2}{\partial t}(s,t_0))^2+(\frac{\partial x_3}{\partial t}(s,t_0))^2=k^2.$$

Q.E.D.

Theorem 3.4. Let $S(s,t)$ be a q-surface pencil which is given as (3.1), where

$$\begin{aligned} x_1(s,t) &= \varphi_1(s)\psi_1(t),\\ x_2(s,t) &= \varphi_2(s)\psi_2(t),\\ x_3(s,t) &= \varphi_3(s)\psi_3(t), \end{aligned}$$

$0\leq s\leq L$, $0\leq t\leq T$, $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$, $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ are C^1 functions and $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$ are not identically zero. Then, $\alpha(s)$ is a parametric and a geodesic curve on the surface $S(s,t)$ if and only if

$$\psi_1(t_0)=\psi_2(t_0)=\psi_3(t_0)=0, \tag{3.4}$$

$$(\varphi_3(s)\psi_3'(t_0))^2+(\varphi_2(s)\psi_2'(t_0))^2=k^2, \tag{3.5}$$

where $k\in\mathbb{R}$, $k\neq 0$.

Proof. We know that $\alpha(s)$ is a parametric curve on the surface $S(s,t)$ if and only if

$$x_1(s,t_0)=x_2(s,t_0)=x_3(s,t_0)=0.$$

Since the functions $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$ are not identically zero, we get $\alpha(s)$ is a parametric curve on the surface $S(s,t)$ if and only if

$$\psi_1(t_0)=\psi_2(t_0)=\psi_3(t_0)=0.$$

Assuming $\alpha(s)$ to be a parametric curve on the surface $S(s,t)$, simple calculations give us

$$\mu_2(s,t_0)=-\varphi_3(s)\psi_3'(t_0) \text{ and } \mu_3(s,t_0)=\varphi_2(s)\psi_2'(t_0).$$

From Corollary 3.3, we obtain $\alpha(s)$ is a geodesic curve on the surface $S(s,t)$ if and only if

$$(\varphi_3(s)\psi_3'(t_0))^2+(\varphi_2(s)\psi_2'(t_0))^2=k^2,$$

where $k\in\mathbb{R}$, $k\neq 0$. Q.E.D.

Theorem 3.5. Let $S(s,t)$ be a q-surface pencil which is given as (3.1), where

$$\begin{aligned} x_1(s,t) &= \varphi_1(s)+\psi_1(t),\\ x_2(s,t) &= \varphi_2(s)+\psi_2(t),\\ x_3(s,t) &= \varphi_3(s)+\psi_3(t), \end{aligned}$$

$0 \leq s \leq L$, $0 \leq t \leq T$, $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$, $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ are C^1 functions and $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$ are not identically zero. Then, $\alpha(s)$ is a parametric and a geodesic curve on the surface $S(s,t)$ if and only if

$$\psi_1(t_0) = \psi_2(t_0) = \psi_3(t_0) = 0, \tag{3.6}$$

$$(\psi_2'(t_0))^2 + (\psi_3'(t_0))^2 = k^2, \tag{3.7}$$

where $k \in \mathbb{R}$, $k \neq 0$.

Proof. We know that $\alpha(s)$ is a parametric curve on the surface $S(s,t)$ if and only if

$$x_1(s,t_0) = x_2(s,t_0) = x_3(s,t_0) = 0.$$

Since the functions $\varphi_1(s)$, $\varphi_2(s)$, $\varphi_3(s)$ are not identically zero, we get $\alpha(s)$ is a parametric curve on the surface $S(s,t)$ if and only if

$$\begin{aligned} \varphi_1(s) &= -\psi_1(t_0) \neq 0, \\ \varphi_2(s) &= -\psi_2(t_0) \neq 0, \\ \varphi_3(s) &= -\psi_3(t_0) \neq 0. \end{aligned}$$

Assuming $\alpha(s)$ to be a parametric curve on the surface $S(s,t)$, simple calculations give us

$$\mu_2(s,t_0) = -\psi_3'(t_0) \text{ and } \mu_3(s,t_0) = \psi_2'(t_0).$$

From Corollary 3.3, we obtain $\alpha(s)$ is a geodesic curve on the surface $S(s,t)$ if and only if

$$(\psi_2'(t_0))^2 + (\psi_3'(t_0))^2 = k^2,$$

where $k \in \mathbb{R}$, $k \neq 0$. Q.E.D.

4 q-Surface pencil with a common asymptotic curve

Theorem 4.1. Let $S(s,t)$ be a q-surface pencil which is given as (3.1) and $\alpha(s)$ be a parametric curve on the surface $S(s,t)$. Then, $\alpha(s)$ is an asymptotic curve on the surface $S(s,t)$ if and only if

$$k_1(s)\frac{\partial x_3}{\partial t}(s,t_0) = k_2(s)\frac{\partial x_2}{\partial t}(s,t_0), \tag{4.1}$$

where $k_1(s)$, $k_2(s)$ are the quasi curvatures of the curve $\alpha(s)$.

Proof. We know that a parametric curve $\alpha(s)$ on a surface is an asymptotic curve if and only if the normal vector field $m(s,t_0)$ of the surface along the curve $\alpha(s)$ is perpendicular to the principal normal vector field $\mathbf{N}(s)$ of the curve $\alpha(s)$, that is, $\langle m(s,t_0), \mathbf{N}(s)\rangle = 0$. Also, we know that $\langle m(s,t_0), \mathbf{T}(s)\rangle = 0$, [4]. By differentiating both sides of this equation, one can get

$$\left\langle \frac{\partial m}{\partial s}(s,t_0), \mathbf{T}(s) \right\rangle = 0. \tag{4.2}$$

So, this equality becomes the necessary and sufficient conditon for $\alpha(s)$ to be an asymptotic curve on the surface. The normal vector field $m(s,t_0)$ of the surface $S(s,t)$ along the curve $\alpha(s)$ is

$$m(s,t_0) = \mu_1(s,t_0)\mathbf{t}_q(s) + \mu_2(s,t_0)\mathbf{n}_q(s) + \mu_3(s,t_0)\mathbf{b}_q(s), \tag{4.3}$$

where

$$\begin{aligned}
\mu_1(s,t_0) &= \frac{\partial x_3}{\partial t}(s,t_0)\frac{\partial x_2}{\partial s}(s,t_0) - \frac{\partial x_2}{\partial t}(s,t_0)\frac{\partial x_3}{\partial s}(s,t_0),\\
\mu_2(s,t_0) &= -(1+\frac{\partial x_1}{\partial s}(s,t_0))\frac{\partial x_3}{\partial t}(s,t_0) - \frac{\partial x_3}{\partial t}(s,t_0)\frac{\partial x_1}{\partial s}(s,t_0),\\
\mu_3(s,t_0) &= (1+\frac{\partial x_1}{\partial s}(s,t_0))\frac{\partial x_2}{\partial t}(s,t_0) - \frac{\partial x_1}{\partial t}(s,t_0)\frac{\partial x_2}{\partial s}(s,t_0).
\end{aligned}$$

Since $\alpha(s)$ is a parametric curve on the surface $S(s,t)$,

$$x_1(s,t_0) = x_2(s,t_0) = x_3(s,t_0) = 0.$$

Using this fact, we get

$$\begin{aligned}
\mu_1(s,t_0) &= 0,\\
\mu_2(s,t_0) &= -\frac{\partial x_3}{\partial t}(s,t_0),\\
\mu_3(s,t_0) &= \frac{\partial x_2}{\partial t}(s,t_0),
\end{aligned}$$

so we find

$$m(s,t_0) = -\frac{\partial x_3}{\partial t}(s,t_0)\mathbf{n}_q(s) + \frac{\partial x_2}{\partial t}(s,t_0)\mathbf{b}_q(s). \tag{4.4}$$

Then, one can easily get

$$\begin{aligned}
\frac{\partial m}{\partial s}(s,t_0) &= (k_1(s)\frac{\partial x_3}{\partial t}(s,t_0) - k_2(s)\frac{\partial x_2}{\partial t}(s,t_0))\mathbf{t}_q(s)\\
&\quad -(\frac{\partial^2 x_3}{\partial s \partial t}(s,t_0) + k_3(s)\frac{\partial x_2}{\partial t}(s,t_0))\mathbf{n}_q(s)\\
&\quad +(\frac{\partial^2 x_2}{\partial s \partial t}(s,t_0) - k_3(s)\frac{\partial x_3}{\partial t}(s,t_0))\mathbf{b}_q(s).
\end{aligned}$$

From condition (4.2), we derive

$$\begin{aligned}
\left\langle \frac{\partial m}{\partial s}(s,t_0), \mathbf{T}(s) \right\rangle &= 0 \Leftrightarrow \left\langle \frac{\partial m}{\partial s}(s,t_0), \mathbf{t}_q(s) \right\rangle = 0\\
&\Leftrightarrow \quad k_1(s)\frac{\partial x_3}{\partial t}(s,t_0) - k_2(s)\frac{\partial x_2}{\partial t}(s,t_0) = 0\\
&\Leftrightarrow \quad k_1(s)\frac{\partial x_3}{\partial t}(s,t_0) = k_2(s)\frac{\partial x_2}{\partial t}(s,t_0).
\end{aligned}$$

Same result can also be obtained as follows:

$$\begin{aligned}
\langle m(s,t_0), \mathbf{N}(s)\rangle &= 0 \Leftrightarrow \langle m(s,t_0), k_1(s)\mathbf{n}_q(s) + k_2(s)\mathbf{b}_q(s)\rangle = 0\\
&\Leftrightarrow \quad k_1(s)\langle m(s,t_0), \mathbf{n}_q(s)\rangle + k_2(s)\langle m(s,t_0), \mathbf{b}_q(s)\rangle = 0\\
&\Leftrightarrow \quad -k_1(s)\frac{\partial x_3}{\partial t}(s,t_0) + k_2(s)\frac{\partial x_2}{\partial t}(s,t_0) = 0\\
&\Leftrightarrow \quad k_1(s)\frac{\partial x_3}{\partial t}(s,t_0) = k_2(s)\frac{\partial x_2}{\partial t}(s,t_0).
\end{aligned}$$

Consequently, $\alpha(s)$ is an asymptotic curve on the surface $S(s,t)$ if and only if

$$k_1(s)\frac{\partial x_3}{\partial t}(s,t_0) = k_2(s)\frac{\partial x_2}{\partial t}(s,t_0).$$

Q.E.D.

5 Offset q-surface pencil with a common geodesic and a common asymptotic curve

We can define offset q-surface pencil similar to the definition of offset surface pencil in [9], as follows:

Definition 5.1. Let $S(s,t)$ be a q-surface pencil which is given as (3.1), $m(s,t)$ be its unit normal vector field and $\alpha(s)$ be a parametric curve on the surface $S(s,t)$. The surface defined by

$$\overline{S}(s,t) = S(s,t) + dm(s,t) \tag{5.1}$$

is called an offset q-surface pencil, where $d \in \mathbb{R}$, $d \neq 0$.

Theorem 5.2. Let $\overline{S}(s,t) = S(s,t) + dm(s,t)$ be an offset q-surface pencil. The parametric curve $\beta(s) = \alpha(s) + dm(s,t_0)$ on $\overline{S}(s,t)$ is a geodesic if and only if

$$\begin{aligned}
&\frac{\partial^2 x_3}{\partial s\partial t}(s,t_0)(k_2(s)k_3(s) + 2k_1(s)) + \frac{\partial^2 x_2}{\partial s\partial t}(s,t_0)(k_1(s)k_3(s)\\
&\quad -2k_2(s)) + \frac{\partial x_3}{\partial t}(s,t_0)k_1'(s) - \frac{\partial x_2}{\partial t}(s,t_0)k_2'(s) = 0,\\
&\frac{\partial^2}{\partial s^2}(\frac{\partial x_3}{\partial t}(s,t_0)) + 2k_3(s)\frac{\partial^2 x_2}{\partial s\partial t}(s,t_0) + (k_1(s)k_2(s) + k_3'(s))\\
&\times\frac{\partial x_2}{\partial t}(s,t_0) - ((k_1(s))^2 + (k_3(s))^2 + \frac{c}{d})\frac{\partial x_3}{\partial t}(s,t_0) = \frac{k_1(s)}{d},\\
&\frac{\partial^2}{\partial s^2}(\frac{\partial x_2}{\partial t}(s,t_0)) - 2k_3(s)\frac{\partial^2 x_3}{\partial s\partial t}(s,t_0) + (k_1(s)k_2(s) + k_3'(s))\\
&\quad\times\frac{\partial x_3}{\partial t}(s,t_0) - (2(k_2(s))^2 + \frac{c}{d})\frac{\partial x_2}{\partial t}(s,t_0) = -\frac{k_2(s)}{d},
\end{aligned} \tag{5.2}$$

where $k_1(s)$, $k_2(s)$, $k_3(s)$ are the quasi curvatures of the curve $\alpha(s)$ and $c \in \mathbb{R}$, $c \neq 0$.

Proof. Let $\overline{m}(s,t)$ be the normal vector field of $\overline{S}(s,t)$ and $\overline{\mathbf{N}}(s)$ be the principal normal vector field of the curve $\beta(s)$. Then, $\beta(s)$ is a geodesic curve on the surface $\overline{S}(s,t)$ if and only if the normal vector field $\overline{m}(s,t_0)$ of the surface $\overline{S}(s,t)$ along the curve $\beta(s)$ is parallel to $\overline{\mathbf{N}}(s)$, that is, $\overline{m}(s,t_0)\|\overline{\mathbf{N}}(s)$. Since $\overline{m}(s,t_0) = \pm m(s,t_0)$ and $\overline{\mathbf{N}}(s) = \frac{\beta''(s)}{\|\beta''(s)\|}$, we get $\beta(s)$ is a geodesic curve on the surface $\overline{S}(s,t)$ if and only if $m(s,t_0)\|\beta''(s)$. We know that

$$m(s,t_0) = -\frac{\partial x_3}{\partial t}(s,t_0)\mathbf{n}_q(s) + \frac{\partial x_2}{\partial t}(s,t_0)\mathbf{b}_q(s),$$

So, the necessary and sufficient condition for $\beta(s)$ to be a geodesic curve on the surface $\overline{S}(s,t)$ becomes

$$\beta''(s) = -c\frac{\partial x_3}{\partial t}(s,t_0)\mathbf{n}_q(s) + c\frac{\partial x_2}{\partial t}(s,t_0)\mathbf{b}_q(s),$$

where $c \in \mathbb{R}$, $c \neq 0$. One can easily see that

$$\beta''(s) = \alpha''(s) + d\frac{\partial^2 m}{\partial s^2}(s,t_0). \tag{5.3}$$

Simple calculations give us

$$\alpha''(s) = k_1(s)\mathbf{n}_q(s) + k_2(s)\mathbf{b}_q(s),$$

and

$$\begin{aligned}\frac{\partial^2 m}{\partial s^2}(s,t_0) \quad = \quad & \left[\begin{array}{c} \frac{\partial^2 x_3}{\partial s\partial t}(s,t_0)(k_2(s)k_3(s)+2k_1(s)) + \frac{\partial^2 x_2}{\partial s\partial t}(s,t_0)(k_1(s)k_3(s) \\ -2k_2(s)) + \frac{\partial x_3}{\partial t}(s,t_0)k_1'(s) - \frac{\partial x_2}{\partial t}(s,t_0)k_2'(s) \end{array}\right]\mathbf{t}_q(s) \\ & - \left[\begin{array}{c} \frac{\partial^2}{\partial s^2}(\frac{\partial x_3}{\partial t}(s,t_0)) + 2k_3(s)\frac{\partial^2 x_2}{\partial s\partial t}(s,t_0) + (k_1(s)k_2(s)+k_3'(s)) \\ \times\frac{\partial x_2}{\partial t}(s,t_0) - ((k_1(s))^2+(k_3(s))^2)\frac{\partial x_3}{\partial t}(s,t_0) \end{array}\right]\mathbf{n}_q(s) \\ & + \left[\begin{array}{c} \frac{\partial^2}{\partial s^2}(\frac{\partial x_2}{\partial t}(s,t_0)) - 2k_3(s)\frac{\partial^2 x_3}{\partial s\partial t}(s,t_0) + (k_1(s)k_2(s)+k_3'(s)) \\ \times\frac{\partial x_3}{\partial t}(s,t_0) - 2(k_2(s))^2\frac{\partial x_2}{\partial t}(s,t_0) \end{array}\right]\mathbf{b}_q(s).\end{aligned}$$

Putting these values in equality (5.3), we obtain the necessary and sufficient condition for $\beta(s)$ to be a geodesic curve on the surface $\overline{S}(s,t)$ as indicated in the theorem. Q.E.D.

Theorem 5.3. Let $\overline{S}(s,t) = S(s,t) + dm(s,t)$ be an offset q-surface pencil. The parametric curve $\beta(s) = \alpha(s) + dm(s,t_0)$ on $\overline{S}(s,t)$ is an asymptotic curve if and only if

$$\begin{aligned} & \left\{k_2(s) + d\left[\begin{array}{c} \frac{\partial^2}{\partial s^2}(\frac{\partial x_2}{\partial t}(s,t_0)) - 2k_3(s)\frac{\partial^2 x_3}{\partial s\partial t}(s,t_0) + (k_1(s)k_2(s)+k_3'(s)) \\ \times\frac{\partial x_3}{\partial t}(s,t_0) - 2(k_2(s))^2\frac{\partial x_2}{\partial t}(s,t_0) \end{array}\right]\right\}\frac{\partial x_2}{\partial t}(s,t_0) \\ = \quad & \left\{k_1(s) - d\left[\begin{array}{c} \frac{\partial^2}{\partial s^2}(\frac{\partial x_3}{\partial t}(s,t_0)) + 2k_3(s)\frac{\partial^2 x_2}{\partial s\partial t}(s,t_0) + (k_1(s)k_2(s)+k_3'(s)) \\ \times\frac{\partial x_2}{\partial t}(s,t_0) - ((k_1(s))^2+(k_3(s))^2)\frac{\partial x_3}{\partial t}(s,t_0) \end{array}\right]\right\}\frac{\partial x_3}{\partial t}(s,t_0), \end{aligned}$$

where $k_1(s)$, $k_2(s)$ and $k_3(s)$ are the quasi curvatures of the curve $\alpha(s)$.

6 Conclusion

Surface pencil with a common geodesic curve is studied by Wang et al. in [16] and surface pencil with a common asymptotic curve is studied by Bayram et al. in [4]. They used the Frenet frame of the curves in their studies. In this work, we consider this subject using the q-frame of the curves. Moreover, we study offset q-surface pencil and give the necessary and sufficient conditions for a parametric curve on an offset q-surface pencil to be a geodesic curve and to be an asymptotic curve on the surface.

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